Learning Mathematics as Language-Games Transitions

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AALBORG UNIVERSITY

LEARNING MATHEMATICS AS LANGUAGE-GAMES TRANSITIONS

by

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Learning Mathematics as Language-Games Transitions

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Abstract I discuss that mathematics among other things is a language (§1). But a language is part of a form of life and a social process, which is why I use the Wittgensteinian notion of language-games (§2). Learning can take place through a negotiation in a dialogue, where I find some important variables (SIMUR-variables): the *S*etting, *I*ntentions, *M*etaphysics, *U*nderlying language, and range of *R*eference of the interacting individuals (§3 & 4). I create a model of learning via these variables. My model reflects learning of school mathematics as being able to move from one's daily language-game (of mathematics), spanned by one set of SIMUR-variables, to the school mathematics language-game, spanned by another set of SIMUR-variables. Now the individuals have developed their SIMUR-variables and are able to participate in both language-games (§5 & 6). Finally, I discuss my model, and what it can add to the discussion of learning (§7).[1]

1 Various languages of mathematics

What is mathematics? This question cannot be answered in a few pages, but I want to focus on activity and language as particular basic characteristics of mathematics. Wittgenstein says that: "Of course, in one sense mathematics is a branch of knowledge, - but still it is also an *activity.*" (1983, p. 227). Pimm states that "mathematics is a language" (1990, p. 2). The word 'is' constructs a metaphor that means that 'mathematics' has some characteristics of the phenomena that characterize a language[2]. Among many things, mathematics is therefore a special kind of activity, that takes place using a language and which also itself becomes a language.

Different cultures have developed different forms of mathematics depending on the needs, ways of thinking, and languages present in these cultures (see for instance Bishop

[1] This paper is a revised version of Dahl (1996). A more elaborate version in Danish can be found in Dahl (1995).

[2] These phenomena are meanihg, symbol, and syntax. Pimm (1990).

(1988)). Here, I speak not only of the mathematics developed in different countries but also of, for instance, Western childrens' informal methods for calculating. Mathematics is an activity that develops and is learnt through a language. The informal languages of mathematics influence the learning process of the formal mathematics in the school - also when they are not considered: "This conflict [in relation to a specific multiplication problem] occurs because the school mathematics culture does not acknowledge or take into account the mathematics practised in the out-of-school culture." (Vithal, 1992, p. 59). A theory of learning mathematics must therefore consider the various mathematics languages, and that different cultures and actions lie behind these.

2 Learning from the perspective of language-games

What does it mean to learn a language? The meaning of a word is not only what the word stands for, but also, and more, its *use* in a language. "Only in the practice of a language can a word have meaning." (Wittgenstein, 1978, p. 344). Wittgenstein invented the concept 'language-game', about which he says: "I shall also call the whole, consisting of language and the actions into which it is woven, the "language-game". . . . Here the term "language-*game*" is meant to bring into prominence the fact that the *speaking* of language is part of an activity, or of a form of life." (1983, §7 & §23). Learning language is more than learning words. You learn a language by putting it into practice, and meaning is present in the social pattern of *its use,* which is also intertwined with other aspects of social life. To learn mathematics is therefore to be able to act in the school mathematics language-game.

I view learning in a classroom from the perspective of negotiation of meaning between the teacher (representing formalized culturally accumulated knowledge) and the pupils (with their own subjective meanings). Through his own actions, the pupil re-constructs the formalized knowledge.

> My thesis is that *learning mathematics* is a transitional process from one mathematics language-game to another, through which the pupils become capable of constructing the mathematics they must learn. Learning mathematics is also to be able to move from one's daily language-game to the school mathematics language-game.

3 Characteristics of mathematics language-games

There cannot be given any formal definition of a language-game and nothing is common for all language-games, says Wittgenstein (1983, §65). What characterizes all language-games is that they, for example pair by pair, have similarities. Wittgenstein calls this family resemblances (1983, §66). The family resemblances within groups of language-games can be pulled out. This means that the family resemblances of a group of

mathematics language-games are the mathematics (reasoning, terms, rules, etc.) common to those language-games. These family resemblances can be uncovered from teaching situations in mathematics where many different mathematics language-games are present at the same time and some of them (for example the pupils') develop.

Between every language-game there will be all kinds of resemblances, big and small. Each possible resemblance lies within an 'important aspect' of this group of language-games. I'm only interessted in what these important aspects look like, since focusing on every resemblance only leads to a jumble of factors. My prime interest is which major factors characterize mathematics language-games.

A language-game is not a *thing* and the important aspects of language-games are not things that exist per se. The important aspects are personal and their content or value is brought into a dialogue by the people.

3.1 Completely understanding - and learning

Before I develop the important aspects I want to discuss an issue I find important in the discussion of learning mathematics.

I will distinguish between two notions. I use the notion 'to understand' in the sense of confirming or agreeing, and to cover what happens when, for example, two individuals know each other and recognize what the other one talks about. I use the notion 'to learn' to cover what goes on in a negotiation when, for example, the pupil increases his knowledge about a topic. Here, I talk about the activity that takes place inside a person when this person 'learns' something. *After* a person has 'learnt' something - he *now* 'understands' this thing. 'Understand' is standstill and 'learn' is activity. It is problematic to make a sharp demarcation between the two notions because they in reality are very closely linked, but I do it to say more about what goes on in a classroom and about the important aspects.

I have an 'ideal thesis' saying that in the ideal case, which only occurs in theory, two people, alike at all the important aspects regarding language-games, will understand each other *completely*. Two individuals can understand each other even though they do not have the same content in these aspects. But they do not understand each other *completely.* For two individuals to be able to understand each other completely, the value of their important aspects *must be* the same. If one individual should learn something from another individual, the value of their important aspects *should not* be exactly the same. Instead there should be clarity about the participating people's content in their important aspects, otherwise a negotiation cannot take place, because this (I will show later) will lead to conflict: "In cases of conflict, the accomplishment of intersubjective meanings taken as mathematical meanings becomes problematic." (Voigt, 1994, p. 281).

The interesting things are: Which aspects are so important for a language-game that a lack of *clarity* about them will lead to breakdowns of the dialogue and cause the two people from their respective language-games not to *learn* from each other? Which aspects must be *similar,* if some degree of *understanding* is to take place?

A language-game developes in a context. The important aspects inside a person that to a great extend influence the language-game in which the person participates, depend on the context. To find these aspects it is necessary to look at real classroom situations and try here to derive the aspects. If some of these are different in two language-games, the two language-games are different. A metaphor of this is a tent and the ropes that span the tent. If one rope is moved, the tent looks different compared to before. In the same way, I choose to look at these aspects and say that they *span the language-games.*

Learning depends on developing a language-game between individuals where the risks of misunderstandings are very small. To develop such a language-game it is necessary that the individuals in the interaction *are clear about* their *intentions* with the dialogue, *range of references, underlying language, metaphysical views* and which attitudes they have towards the physical *setting* in which they are placed. A lack of clarity or correspondance about just one of these aspects does not mean that they cannot communicate at all. But a lack of clarity in just one aspect is enough for misunderstandings appear. The range of misunderstandings and their number depend on in how many aspects there is a lack of clarity, and the extension hereof. I will focus on that misunderstandings appear when there is a lack of clarity or similarity between the individuals engaged in a dialogue in some of the above-mentioned aspects.

Now I will explain the 5 aspects.

4 Development of the important aspects of mathematics language-games

I pursue the idea mentioned above about looking for the aspects in a context of teaching mathematics. I will use empirical material to illustrate my thesis about learning mathematics, and so try to find which aspects are the 'main aspects'.

I will use transcriptions from a teaching period in a Danish 3rd grade class from Alrø & Skovsmose (1993)[3] and transcriptions from a teaching period on probability in a German 6th grade class from Steinbring (1991). A few other sourses will also be used. I chose these papers because of their long transcripts, which make it possible, independently of the writers' own interpretations, to get an idea about what has happened in the teaching episodes. Sometimes I use their conclusions and sometimes I reinterpret them.

4.1 The importance of the SETTING

The physical environment, setting, or frame of a dialogue and how the individuals in the interaction respond to this environment influence the language-game developed. A teacher will for example very often refuse to move the lecture to the schoolyard, even when

[3] The paper is in Danish, so every time I quote it, it is my own translation.

the weather is nice, because this environment invites cosiness and little seriousness, which is incompatible with a teaching situation of mathematics. If one person in the interaction feells more 'at home' in the setting than the others, this will also have an effect. The language-game changes even between the same two people depending on whether they talk in a church, in a pub, or in a classroom. The language-game does however not change if both of them have the same 'feelings' for the three places.

It is not always the physical environment itself that directly influences the individuals, as it is the case, when individuals get tired on a hot day. More often it is the individuals' attitudes towards the environment, and what is socially accepted as customs, that affect the individuals. A teaching situation where the setting is not the same for the individuals is when the teacher and the pupil communicate through a telephone or television.

From now on, the variable 'S' stands for the outer physical context, frame, environment, or setting of a dialogue. The words just listed are metaphors for the variable and I will use them synonymously. This is also the case for the future variables. When I talk about the setting around person number 1, and his attitudes toward this, I call this S_1.

I understand the setting in two different ways. First, as the 'external setting', namely the school, the classroom, the other pupils, their clothes, the weather, time of the day, time of the week and so on. Second, as the 'internal setting', which is when the teacher and the pupils are tied up with how a text book expounds the mathematics. This decides to a certain degree how the teacher evaluates the pupils' questions. I will concentrate on the former. To find adequate examples of this, I use investigations which do not examine how an interaction *inside* a classroom takes place.

4.1.1 Different rules in different settings

Carraher et al. (1985) investigate the informal mathematics of Brasilian children and compare this with their formal mathematics. The children are first exposed to "informal tests", in their natural working situation, on streetcorners, and then "formal tests" in a context "similar to the school setting." (p. 24). The example shows many things but an important result is that when the children are in a home-context, they solve the problems with the help of their own home-made methods. But when they meet the same kind of problems in a setting that looks like the setting of a school teaching situation they change their method for solving the problems. When children enter an S which looks like the S of a school mathematics teaching situation "the children try to follow, without success, school-prescribed routines." (p. 27). The setting determines which kind of action the pupils expect of themselves. Also Christiansen talks about this: "The setting constricts the reading of the context, the relation between context and mathematics, and the mathematical activity." (1996, p. 252).

The custom behaviours inside the setting of the dialogue play a role in how the individual acts. If the individuals ascribe different rules to the same setting, and there is not clarity about this, it can lead to a conflict between the individuals inside the setting. In the example of Carraher et al., there is a difference between which rules the children follow in the various settings. This meant that the observer was astonished - why do they not calculate the problems they meet in the school context the same way they do at the market? I think the children are surprised by the thought that it is *strange,* that they did not use their informal methods in the school context.

4.2 The importance of INTENTIONS

The individuals' intentions are also important for the dialogue. Two pupils do not reach a result if, for instance, one of them tries to learn the subject because it is interesting, and the other one only wants to pass an exam and then forget everything. Nor does an individual necessarly have only one intention for participating in a dialogue. For instance a teacher can both intend to earn money and feel a vocation for being a teacher. The pupils might also have a multiplicity of intentions for going to school. Maybe they want to make the teacher happy or maybe they want to learn something. The list could be longer but this kind of intentions is not what I want to address. Instead I concentrate on 'private intentions', for instance what the teacher wants the pupils to learn in a certain lesson, or which intentions lie behind a particular question. The 'private intentions' for the pupils might be whether they only want to guess the answers, or instead want to take an active part in the teaching.

From now on the variable 'I' stands for the intention, goal, or plan, and what an individual sees as his 'job' in a dialogue. When I talk about the intention of person number 1, I call this I_1.

4.2.1 When the intentions are not shared - but hidden

Alrø & Skovsmose (1993) have investigated the communication in a mathematics teaching situation whose purpose is invisible to the pupils. They refer to the project: "Newspapers, how much do they fill?". This question is an introduction to 'area and volume'. To 'fill' is an ambiguous word that is used in many contexts and therefore in many different language-games. To 'fill' can for instance describe how much a heap of newspapers 'fills' (volume) and can descibe how much the newspapers 'fill' when they are spread out on a table (area). In the beginning of the lesson the teacher wants to take advantage of this ambiguity and play with this. The teacher confuses the pupils because he not only plays with 'fill's' ambiguity, but he especially plays with synonyms of the word that give associations to volume and succeedingly he wants the pupils to learn about area. For example he talks about how thick newspapers are. The reason the pupils do not quit the dialogue is that "they probably know that if they follow the teacher then everything will be all right and they will reach the knowledge they have to learn." (p. 15). Therefore the pupils' intentions are to find out where the teacher is going. The pupils conclude that when this is not being told to them, when they ask, they have to guess. The pupils want to take active part in the lesson, but because of the teacher's introduction, this intention can only be realized through guessing. "The dialogue shows that the teacher has an idea or an

intention which he follows, while the pupils have to use guessing to follow the dialogue. This results in two different ways of communication." (p. 17). Inside the language-game the teacher and the pupils develop, they talk with crossed purposes.

This does not mean that a dialogue where the participants do not have the same intentions never will be a success. But when the pupils' I is to guess the teacher's I, and the teacher's I is that the pupils should ***not guess*** but learn about area in a certain (contructiviste) way, there is a difference in the I's of the dialogue. More important, there is a difference between the I the participants expect the other participants to have, and what the other participants have in reality. When the teacher's I is hidden from the pupils, the pupils think that the teaching situation is a 'guessing contest', while the teacher sees the pupils' I to be "that the pupils should develop and invent mathematics through a process of working and acting, rather than they should be informed about the notions." (p. 11)[4]. There is an extensive lack of clarity about the I's. There cannot be build a language-game with a direct flow of communication.

4.3 The importance of METAPHYSICAL VIEWS

The metaphysics of individuals, their view about human beings and their opinions about what is right, what works, and who is an authority are also important. The pupil's evaluation of whether he has learned anything in a lesson has to do with what he finds is real mathematics. If, to him, real mathematics is something in an imaginary world, he might feel that he has learnt a lot if he has learnt to prove theorems. If the teacher has made him play with dice to make him 'see' some rules of mathematics, he would describe this as a waste of time. This aspect has to do with which understanding of mathematics he considers true, and the attitude with which a person enters a classroom. It is reflected in what is said 'between the lines' and is a kind of meta-content of the dialogue.

If two individuals converse and both agree that the opinions of one of them have more value than the opinions of the other (for example a teacher or an officer) this affects the language-game they develop. This language-game will be different from the language-game they develop if the opinions are 'equal' or if they disagree about whose opinions have the most value. Among the participants there must be clarity about this if they are to learn from each other. The un-said must be as if it has been said.

From now on the variable 'M' stands for the people's metaphysical view, their 'philosophy', their unconscious way of 'seeing' and interpretating, their faith, what they see as given, and what they expect to be between the lines in a dialogue. When I talk about the metaphysical views of person number 1, I call this M_1.

[4]This is connected with a view that a true act emergs from the individuals' own motivations, and human behavior which has not its roots in an intention is not an act. (Mellin-Olsen, 1987, p. 30)

4.3.1 When different people each have their own understanding of mathematics

All guesses on the meaning of 'fill' are in the beginning in **Alrø & Skovsmose (1993)** equal. But at some point the teacher changes strategy and now looks for a particular meaning of 'fill'. The teacher does not announce this shift in perspective. The teacher has one particular route he feels is the right one and this is hidden from the pupils. It now becomes difficult to learn through this language-game. The dialogue only *appears* to be open because in reality the teacher does not listen to the pupils' suggestions. If this happens repeatedly it might result in a metaphysic of the pupils saying that "the teacher is always right and no matter what we think, it is always the teacher who sets the agenda and decides what is right and wrong, important and not important. . . . In mathematics there is always one and just one true answer: The pupils' "philosophy of mathematics" might therefore be a reflection of the teacher's communication." (p. 18-19). The pupils are left with a philosophy - M_p - saying that the teacher is an authority to be followed blindly. From the pupils' perspective, the language-game developed is subject to a kind of 'authority criterion'. The teacher, on the contrary, has the metaphysic - M_t - that the pupils should construct the mathematical notions themselves, and therefore he tries to develop a language-game subject to a kind of 'constructivism criterion'. When M_p is different from M_t and furthermore there is a lack of clarity about this, the participants in the language-game developed will talk with crossed purposes and learning cannot take place.

Generally, M is a kind of fundamental criterion for what should or could be done in the context of the language-game. The other variables have each particular rules and criteria.

4.4 The importance of the UNDERLYING LANGUAGE

It is also important with which language the participants enter the dialogue. I am not talking about language-games here. Language-games are not something we *have*, but something we *do*. But the rules for the ways an individual uses the words and which meaning a particular composition of words (sentences) has is fixed inside the home language of the individual. The 'underlying language' is - so to speak - "language minus speech" (Barthes, 1990, p. 14). It is the stiffer underlying 'non-living' and non-playing structures, the sense of the words. This must be the same if the participants are to understand each other, and it should be known which kind of underlying language the other participants possess, if learning is to be possible.

From now on the variable 'U' stands for a person's underlying language entering a dialogue. When I talk about the underlying language of person number 1, I call this U_1.

4.4.1 When the same word have different meanings

In **Alrø & Skovsmose (1993)** the general concept 'fill' is used in different contexts, and therefore in different language-games. In the teaching situation 'fill' is sometimes used meaning area, sometimes meaning volume. 'Fill' is thus an ambiguous word, which in particular contexts becomes unambiguous. It is ambiguous in the beginning because the teacher asks for all kinds of meanings of the word 'fill'. Later the teacher tries to make the pupils construct *the* meaning of the word 'fill'. "What then does it mean to fill?" (p. 16), he asks guidingly - as if 'fill' has only *one* meaning. Before this question, the teacher has given the pupils hints that gave associations to volume. Naturally the pupils now guess: "It might fill a space." (p. 16). This is a good guess seen from their perspective, as the teacher has given the pupils hints in this direction. But a pupil "seems to quit the dialogue completely when the teacher asks whether a newspaper can 'fill' any other way. Actually, he has just offered a such suggestion, but the teacher rejected it." (p. 16). The problem with 'fill's' ambiguity is seen when the teacher prefers one particular meaning of 'fill' in the 'context of guessing' the teacher has built up with the pupils, when a moment before all meanings of 'fill' were equal. The teacher wants the pupils to be part of the school mathematics context, where 'fill' mostly is unambiguous. The pupils' problems are mainly not caused by 'fill's' ambiguous nature, but rather by its unambiguous nature.

If I regard 'fill' as a sign[5], 'fill' is of arbitrary nature because there is not any coherence between 'fill' as a signifier - made up by 4 letters - and what it means to 'fill'. A convention and a social accept once among the users establish the coherence. Thus, 'fill' and 'fill' are two different signs - two different words - with the same signifiers, but different signifieds. The teacher knows both the words; both when 'fill' is ambiguous (now noted as '*fill*') and when it is unambiguous (now noted as 'FILL'). The pupils only know '*fill*'. 'FILL' is not really present in the classroom because only the teacher knows this meaning. It is not socially accepted to connect the signifier 'fill' with the meaning of 'FILL'. Only *fill* is actually present in the classroom.

4.4.2 When different words have the same meaning

In the episodes in **Steinbring (1991),** the teaching has to a great extent to do with naming various concepts. The essential thing is the notation. In the first episode, the teacher asks the pupils to mention all possible outcome from a toss of a die. He evaluates their answers as correct and then presents them with a new word that is meant to cover the meaning of their words: "8 T.: Right! And this set, we are going to call this the fundamental set.". Here, it seems that different ways of formulating a thing have exactly the same meaning. At least their meanings are presented as equal because the teacher only presents what happens as naming. In reality it is an *abstraction,* which I return to later. Then the teacher asks the pupils to write down all the subsets of the fundamental set, which

[5]Saussure sees a sign as something, that is composed of a signifier, which is the symbol or the appearance, and of a signified, which is the content or meaning of the sign. Barthes (1990).

he names 'events'. The teacher's focus on naming things and on writing in a certain unambiguous way, make the pupils (in line 47) to think that the most important thing is the notation - the 'right' vocabulary and the 'right' use of this. If the pupils are not able to switch to the vocabulary the teacher presents to them, the dialogue will break down, which it also does in Steinbring's example.

4.5 The importance of the range of REFERENCES

Also the background knowledge, the experience - the individuals' range of reference - is important. This variable is part of the etnomathematical program because here it is said that mathematics is developed differently in different cultures. For example, D'Ambrosio writes that ethnomathematics is "the mathematics which is practised among identifiable cultural groups, such as national-tribal societies, labour groups, children of a certain age." (1985, p. 45).

An individual has a knowledge, a home-context, experiences, and needs that colour how this individual 'sees' the world. A difference in the range of references between two individuals make them perceive things differently. This can easily lead to misunderstandings and breakdowns of the dialogue. The dialogue's background must be known or shared.

From now on the variable 'R' stands for the range of references, that is, the background knowledge, experiences, and horizon of a dialogue. When I speak of the range of references for person number 1, I call this R_1.

R is often different from M because even though an individual primarily thinks and acknowledges through a language (Mellin-Olsen, 1987, s.77), which has to do with R, a person's metaphysical views primary do not depend on a language. A metaphysical view is a 'thing' and an acknowledgement that might have taken place (maybe unconsciously) outside a language. What is said between the lines in a dialogue will often be acknowledged unconsciously outside a language through, for example, facial expressions or ostensive definitions. Another difference between R and M is the way they can be changed. An R can be caught, understood, debated, and discussed, which an M cannot. An M can be incomprehensible as a notion but it appears and can be exhibited. An M can be changed but is more irrational, which means that M, contrary to R, very often is not changed through the conscious thinking via a language.

4.5.1 When different people refer to different knowledge

In **Steinbring (1991)**, there is a difference between the teacher's knowledge and that of the pupils. This gives problems when the teacher introduces a new mathematical concept, which demands a change in the pupils' background understandings of the mathematical concepts' nature. The teacher already masters and understands the knowledge the pupils are about to learn. The teacher makes the pupils talk about the outcome of the throw of a die: "less than one" (line 48) to establish a connection between the new subject and

what the pupils already know. Here, the teacher probably has experiences that tell him that the pupils know that a throw of a die cannot show a number of eyes less than 1. But the pupils resign from the dialogue because this throw cannot happen. The teacher thinks *abstractly.* The pupils, on the contrary, think of the event as something that can or cannot happen - they think *concretely.* This reveals that there is a difference in the knowledge to which each of them refers. This is even more interesting when the teacher actually thinks that the pupils' answers reflect a knowledge of the pupils that corresponds to the teacher's own, namely that a set can be empty. In reality, the pupils' statements reflect another range of references, namely that a toss of a die only can result in an outcome between 1 and 6 eyes. This confusion around R leads to a breakdown of the dialogue.

5 The SIMUR-model

The five variables S, I, M, U, and R, are *the important aspects* reffered to on page 3. I name the important aspects 'the SIMUR-variables', which underlines that they are not static but develop and change through time. I will now talk about their mutual relationship and argue for their importance for which language-game will come up. I also talk about the SIMUR-variables in relation to individuals and to language-games.

An individual *embraces* SIMUR-variables, which means that the SIMUR-variables are inside the person. Furthermore 'embrace' emphasizes that the variables change through a process where the individual is active and conscious of what happens. A language-game *spans* all the SIMUR-variables that the participants embrace. Each individual has the SIMUR-variables, but it is the variables' different contents or values which the language-game span. A difference in the value of the individuals' SIMUR-variables in two language-games makes the two language-games different. From now on, I will for the sake of convenience mention this as a difference in the SIMUR-variables of two language-games.

Language-games cannot be defined, but I find that the SIMUR-variables are the important aspects of language-games. Therefore they are also important when we talk about learning mathematics as a process of altering variables of one's home-context (mathematics) language-game.

We can look at language-games from two points of view. First, the *outer* reflection, where we concentrate on the differences and similarities between a number of language-games and evaluates these. Second, how it from *within* feels to act in a language-game, and what should be valid if it were to be developed. I do not want to discuss the former point of view, since this has to do with an evaluation of whether the language-game of one group of individuals at one time, and the same group's language-game at another time, are the same. This has to do with an evaluation of the language-games *after* a process of language-game transition has taken place. In this paper my aim is to discuss *how* such a process takes place, wherefore it is more interesting to reflect on which role the SIMUR-variables have in the process of developing a language-game towards a mathematics language-game.

5.1 The importance of the SIMUR-variables for a consideration of language-games from within

The SIMUR-variables must *not* be completely similar, if mathematical knowledge is to be negotiated. It is, however, necessary that the SIMUR-variables are *known and that there is clarity about them.* Some sort of similarity between the SIMUR-variables is nevertheless necessary - especially in the case of U - otherwise it is only possible to smile at each other. In a teaching situation in school, this is generally not so problematic, since pupils and teacher in a classroom already have some kind of shared U - otherwise they should not have been placed in the same class.

A language-game develops as the participants' SIMUR-variables in the interaction develop and change. A language-game where mathematical knowledge can be negotiated and the SIMUR-variables therefore can change, I will call *a direct or an open language-game.* Oppositely, I will call a language-game *a crossed or a closed language-game* if the individuals cannot develop mathematical knowledge within it, because there is not clarity about the SIMUR-variables.

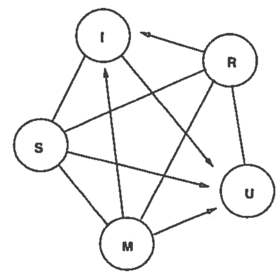
This clarity lacking, they will nevertheless be able to build up a language-game, but this will be a crossed language-game where the communication either breaks down or has a very narrow range. The participants can for instance only talk about the weather. The influence the degree of clarity around a particular SIMUR-variable has in building up a direct language-game, I will not discuss any further. Rather, I will concentrate on that if individuals start to negotiate in a direct language-game they will develop in the direction of being able to play and work with academic mathematics. After some time, one will, from the point of view of *within,* say that *the language-game has developed and changed.* From the *outside* point of view, one will about the same incidence, say that *the individuals have taken part in a transitional process of moving from one (mathematics) language-game to another.* From this point of view, it will look as if the first-observed language-game is different from the last-observed language-game, and that the individuals have moved from one language-game to another.

5.2 The relations between the five variables in a language-game

The SIMUR-model exhibits the complexity of variables critical to the learning process, where none of the variables can be excluded, and all of them are different in nature. U is particulary different, since it is always visible 'in action' within a language-game. S, I, M, and R are more disguised and are dependent on the use of U to become visible.

U is unusual because it is very sensitive in a teaching situation as it is being influenced by all the other variables. Outside a language-game, S, I, M, and R do not affect U, but when a person enters a dialogue, the content of S, I, M, and R will to a certain

degree determine how the U is *used.* I affects U because an individual's intentions for participating in a dialogue to a certain degree affect how he chooses to use his underlying language in the dialogue. A person's R is also important for his U, which is seen when a teacher has some academic knowledge that causes him to present the topic to the pupils with a certain vocabulary. M also affects U, which is seen when a teacher has a certain mathematics philosophy, for example that mathematics is a question about the right notation. The teacher's choice of words reveals his M. Also S affects U because a mathematics teaching setting is dominated by short sentences and a "question - answer - evaluation" dialogue (Lemke, 1990, p. 8), contrary to daily talk. The other variables often affect one another both ways. U affects R since "*language* is the basic thinking-tool" (Mellin-Olsen, 1987, p. 77). A language does not do the learning for us, but it is a tool for it. M is not (always) affected by the U because not all kinds of learning take place through a language. An M is often something that lies outside a language. U does not affect I. An individual can change his I through a rationel thinking process through a language, but to do so, U is only a tool for R. It is through R an individual by thinking determines his I. The same argument goes for S. The figure below shows the relations between the variables. An undirected line indicates influence both ways.



The relations between the important variables of a negotiation in the SIMUR-model.

6 A transitional process from one language-game of mathematics to another from the perspective of U

I will choose one of the SIMUR-variables and describe its transitional process. I choose U because it is the most sensitive, and therefore dependent, variable in the learning proces. I will once more look at the problems of the unambiguous and ambiguous nature of 'fill'. As an example I will describe which process *'fill' should have gone through* for the pupils to make them possess 'FILL'.

'fill' and 'FILL' belong to different language-games. I will consider U's transitional process as a process which mainly consists of a **rulebreaking/ change of criteria,** a **translation/ transfer,** a **generalization/ abstraction,** and an **exemplification/ specification.** Those four I call *the transitional actions,* and look at how learning a U can be described in terms of transitional actions, seen from four different points of views.

Rulebreaking and change of criteria

Rules control and constitute a language. A language's rules depend on a social contract to follow them: "To obey a rule, to make a report, to give an order, to play a game of chess, are *customs* (uses, institutions)." (Wittgenstein, 1983, §199). Mathematics is a language with rules, and has among other things to do with following and using rules (theorems, proofs, refutations). Development in mathematics has to do with proving or disproving existing theorems and making new. A game has rules that give meaning to the game for the individuals who know these ones, but are incomprehensible for those who do not. Participation in *all* sorts of language-games implies following the rules connected to each SIMUR-variable.

After a process of language-game transition, there will be a change in the individual's content of U and new rules will, in accordance with social acceptance, be added to his U.

Words get their meaning from their use, and therefore the pupils must learn the *meaning* behind the teacher's talk and use of 'FILL'. They must realize that it is possible for a word within a particular context to have a unambiguous meaning. The normal (daily) criterion for pupils about words' meaning is that words are ambiguous. They must enculturate that a word can have an unambiguous meaning.

Translation and transfer

A part of learning is a development of one's U and a learning of new words and signs.

For example, the word *'fill'* must be translated or transfered to the word 'FILL'. The words *'fill'* and 'FILL' belong to different language-games, so the two different meanings are due to different criteria. To be able to talk more precisely about U's transition, I will use Saussure's notion of signs. We can now talk about two different forms for the translations of *'fill'*. The translation is from left to right:

	Unambiguous (pupils):	Ambiguous (teacher):
Translation of content:	$\frac{\text{signifier1}}{\text{signified1}}$	$\frac{\text{signifier1}}{\text{signified2}}$
Translation of symbol:	$\frac{\text{'signifier2'}}{\text{signified2}}$	

In the <u>'translation of content'</u> the combination of the letters: 'f-i-l-l' (signifier1) already exists in the pupils' minds before the translation, but their sense of 'fill' is different from the teacher's sense (signifier2). The phase of transfer from the pupils' perspective is therefore that a new meaning is to be transferred to a combination of letters well known to the pupils.

In the <u>'translation of symbol'</u> the pupils know a meaning (signified2), for instance that <u>something might be unambiguous</u>, but they have not yet choosen a suitable combination of letters to cover what they know (hence the quotation marks). Or maybe they have a different combination of letters for this meaning (signifier2). The negotiation of 'fill' between teacher and pupils can take place when there is some sort of 'bridge' between *'fill'* and 'FILL', because either the signified or the signifier is the same.

A problem might arise if none of these cases occur. For example if the pupils think: $\frac{\text{signifier1}}{\text{signified2}}$ and noboby realizes that everybody talks about the same thing. Or if the pupils think: $\frac{\text{signifier2}}{\text{signified1}}$, which means that something *completely* new is being introduced to the pupils. In that case the teacher must start somewhere else in the pupils' knowledge.

Exemplification and specification

Another side of the transition is a specification, just as a word in a daily language is specified when it occurs in a formal language. An example could be the use of 'continuous' instead of 'coherent'. Regarding specification, Zinkernagel says that the notion of mathematics is a "specification of notions we already know from daily language" (1957, p. 11) [my translation]. Specification has to do with crossing from a daily language to a formal mathematical language.

A formal language is an unambiguous language, whence 'FILL' can be said to belong to such a language. *'fill'* belongs to a daily language. Crossing from *'fill'* to 'FILL' can therefore be regarded as a formalization in the form of a specification, where the ambiguous word *'fill'* in some sense is made more exact because *'fill's'* meaning is made particular and definite.

This phase is also an exemplification, because one pulls out a particular meaning from the number of meanings of the ambiguous 'fill' and focuses on this.

Generalization and abstraction

'fill's' transition also involves a generalization or abstraction, that is an opposite process compared to the exemplification and specification. It is a form of abstraction when *'fill'* is removed from its daily use. Its concreteness disappears for the pupils and 'fill' appears as an abstract notion. The abstract and general notions are constructed by the individual: "*Generalizing means Constructing Variables.*" (Dörfler, 1991, p. 84). The theory of Dörfler is namely that the concepts of mathematics can be reconstructed generically from actions in various ways, eg. a negotiation action in a language. Crossing from '*fill*' to 'FILL' can therefore be regarded as a formalization in the form of an abstraction.

As a further example, I will now return to what happened in the teaching episode from Steinbring (1991), here on page 9. Besides giving name to a phenomenon, also an abstraction took place. Before the name giving the set was something *concrete* the pupils had created themselves. After the name giving the 'fundamental set' is born, and now becomes something that exists independently of the 'real' world. The set now belongs to the *imaginary* world of mathematics.[6]

7 Closing

My SIMUR-model states the relations between the important variables of a negotiation of mathematics. The model describe the complexity of a learning process as a language-game transition on five aspects. To describe this process, we must look at this complexity and not just at one of the aspects, as it has been the case in many instances of research in mathematics education.

7.1 SIMUR in relation to some phenomena

What phenomena can we now connect with the SIMUR-model? The SIMUR-model criticize the investigations of others. Alrø & Skovsmose (1993) investigate what happens to the learning process, when the purposes - I - are hidden for the pupils. Christiansen (1996) focuses on the setting's - S - importance for how pupils engage in mathematical activity. Ethnomathematics regard the background of various cultural tribes, societies, labour groups, or children of a certain age - R - and says that different cultures have developed different kinds of mathematics. The teacher in Steinbring's example has understood that mathematics, among other things, is a language and has to do with naming - U. But this one-sided focus has as a result that the pupils do not understand anything, but rather are left with a view of mathematics as something that only has to do with notation. This is not what mathematics is, although it is *among other things* a language.

[6] Here, we could turn to the discussion of 'basic-level effect' from cognitive psychology made by Rosch (see Lakoff (1990, p. 46-57)). Her view is that we can divide words into 3 levels: Superordinate (e.g. animal or furniture), basic level (e.g. dog or chair), and subordinate (e.g. retriever or rocker). The basic level is the level at which most of our knowledge is organized, and it is the first level to enter the lexicon of a language. This view is not directly applicable for a discussion of learning mathematics, but can be used for elaborating exemplification and abstraction . For instance, 'function' or R^2 can be placed on the basic level, 'set theory' or R^n in the superordinate level, and 'composite function' or R^1 as subordinate level. A journey from subordinate level to basic level or from basic level to superordinate level can be regarded as an abstraction, and the opposite direction as an exemplification. Both the abstraction and the exemplification are forms of formalization. This means that FILL belongs to both the subordinate and the superordinate levels, while *fill* belongs only to the basic level.

Therefore, a teacher must focus on mathematics as a language-game, and learning mathematics as language-game transition. This view recognizes that learning mathematics should be seen in relation to the variables S, I, M, U, and R. A learning process is a complex personal process throughout which the individual acts. For practical reasons, it can be necessary to concentrate on just one side of the process. But to be able to give a full description of the learning process, we must take all the aspects into account. Regarding the above-mentioned example of a language-game transition for U, we can say that 'Rulebreaking and change of criteria' of U alone does not tell the whole thing. We first get a deeper understanding of 'Rulebreaking and change of criteria' after looking at the transitional actions for all the SIMUR-variables. The process of language-game transition for the four other variables does not, however, necessarily look like that of U.

The thesis developed through this paper can now be summarized as follows:

> Learning of academic mathematics is a process of language-game transition where the pupil is active and interacts with a teacher within a direct language-game, whereby the pupil develops his SIMUR-variables. Then the pupil is able to participate in the academic mathematics language-game.

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